Time Frequency Localization of Pulse Shaping filters in OFDM/OQAM Systems

Jinfeng Du, and Svante Signell
KTH Department of Electronic, Computer, and Software Systems, Sweden
Email: {jinfeng, srs}@kth.se

Abstract—In this paper we investigate the time frequency localization (TFL) properties of different pulse shapes in OFDM/OQAM systems. Various prototype functions, such as rectangular, half cosine, Isotropic Orthogonal Transfer Algorithm (IOTA) function and Extended Gaussian Functions (EGF) are discussed and implemented on the Matlab/Octave Simulation Workbench [15] for Software Defined Radio by directly discretizing the continuous time model. Simulation results show that pulse shapes with good TFL properties can have near perfect reconstruction.

I. INTRODUCTION

Pulse shaping OFDM/OQAM systems [1]–[3] can achieve smaller combined ISI/ICI without adding any cyclic prefix compared to classic OFDM systems. Various pulse shaping prototype functions with good TFL property have been proposed [5]–[8] and implementation issues based on various filter banks has also been addressed [9]–[11]. Contrary to classic OFDM scheme which modulates each sub-carrier with a complex-valued symbol, OFDM/OQAM modulation carries a real-valued symbol in each sub-carrier and consequently allows time-frequency well localized pulse shape under denser TFL requirement [4]. This enables a very efficient packing of symbols to maximize e.g. the throughput or the interference robustness in the communication link. OFDM/OQAM has already been introduced in the TIA’s Digital Radio Technical Standards [12] and been considered in WRAN (IEEE 802.22) [13].

The transmitted signal in pulse shaping OFDM/OQAM systems can be written in the following analytic form

\[
s(t) = \sum_{n=-\infty}^{+\infty} \sum_{m=0}^{N-1} a_{m,n} g_{m,n}(t) \quad (1)
\]

where \( a_{m,n} \in \mathbb{Z}, m = 0, 1, ..., N - 1 \) denotes the real valued symbols conveyed by the sub-carrier of index \( m \) during the symbol time of index \( n \), and \( g_{m,n}(t) \) represents the synthesis basis which is obtained by the time-frequency translated version of the prototype function \( g(t) \) in the following way

\[
g_{m,n}(t) = e^{j(m+n)\pi/2} e^{j2\pi m n \nu_0 t} g(t-n\tau_0), \quad \nu_0 \tau_0 = 1/2 \quad (2)
\]

A modified inner product for demodulation is defined as follows

\[
\langle x, y \rangle_R = \Re \left\{ \int_{\mathbb{R}} x^*(t) y(t) dt \right\}
\]

where \( \Re \{ \cdot \} \) is the real part operator. It decomposes the lattice points \( g_{m,n} \) into four sub-lattices [5]: 

EO={m even, n odd}, OE={m odd, n even} and OO={m odd, n odd}, as shown in Fig. 1.

Fig. 1. OFDM/OQAM Lattice.

The orthogonality between different sub-lattices is automatically guarantied and is independent of the prototype function as long as this function is even. While inside the same sub-lattice, the orthogonality can be ensured by finding an even prototype function whose ambiguity function \( A_g(\tau, \nu) \) satisfies

\[
A_g(2p\tau_0, 2q\nu_0) = \begin{cases} 
1, & \text{when } (p, q) = (0, 0) \\
0, & \text{when } (p, q) \neq (0, 0) \quad p, q \in \mathbb{Z} 
\end{cases} \quad (3)
\]

Two kinds of realizations of pulse shaping OFDM/OQAM systems are of practical interest as they are very easy to be implemented in a classic OFDM system. Assume \( T \) is the OFDM symbol duration and \( F \) is the inter-carrier frequency spacing, we have \( TF = 1 \) when no cyclic prefix is added. One can either set \( \nu_0 F = F \) and shorten symbol duration [9], [11], or set \( \tau_0 = T \) and double the number of sub-carriers [10]. We use the former approach.

This paper is organized as follows. Section II presents pulse shape prototypes and introduces criteria for TFL property. The continuous and discrete time system models and the direct implementation method are introduced in Section III. Simulation results both on TFL and perfect reconstruction are shown in Section IV and conclusions are drawn in Section V.
II. PULSE SHAPE PROTOTYPES AND TFL

In the following part of this section, several different types of pulse shape functions are presented, followed by the Heisenberg parameter $\xi$ as an indicator for the TFL property.

A. Prototype Functions

1) Rectangular Function:

$$g(t) = \begin{cases} 1, & |t| \leq \frac{\tau_0}{2} \\ 0, & \text{elsewhere} \end{cases}$$ (4)

2) Half Cosine Function:

$$g(t) = \begin{cases} \frac{1}{\sqrt{2}} \cos \frac{\pi t}{\tau_0}, & |t| \leq \frac{\tau_0}{2} \\ 0, & \text{elsewhere} \end{cases}$$ (5)

3) Extended Gaussian Function and IOTA:

$$z_{\alpha,\nu_0,\tau_0}(t) = \frac{1}{2} \sum_{k=0}^{\infty} d_{k,\alpha,\nu_0} \left[ g_\alpha(t + \frac{k}{\nu_0}) + g_\alpha(t - \frac{k}{\nu_0}) \right]$$

$$\sum_{l=0}^{\infty} d_{l,1/\alpha,\tau_0} \cos(2\pi l \frac{t}{\tau_0})$$

where $\tau_0 \nu_0 = \frac{1}{2}, \quad 0.528\nu_0^2 \leq \alpha \leq 7.568\nu_0^2$, $d_{k,\alpha,\nu_0}$ are real valued coefficients and can be computed via the rules described in [5], [9]. This family of functions are named as Extended Gaussian Function (EGF) as they are derived from the Gaussian function $g_\alpha$, which is defined by

$$g_\alpha(t) = (2\alpha)^{1/4} e^{-\pi \alpha^2 t^2}, \quad \alpha > 0$$ (7)

Note that, for EGF and Gaussian functions, their Fourier transforms have the same shape as themselves except for an axis scaling factor [16]

$$\mathcal{F} z_{\alpha,\nu_0,\tau_0}(t) = z_{1/\alpha,\tau_0,\nu_0}(f), \quad \mathcal{F} g_\alpha(f) = g_{1/\alpha}(f)$$ (8)

A special case of EGF, $\zeta(t) = z_{1/\alpha,\tau_0,\nu_0}(t)$, is called Isotropic orthogonal Transform Algorithm (IOTA) Function due to its invariance to Fourier transform $\mathcal{F} \zeta(t) = \zeta(f)$.

B. Ambiguity Function and Heisenberg Parameter

The (auto)-ambiguity function is defined as

$$A_{g}(\tau, \nu) = \int_{\mathbb{R}} e^{-j2\pi \nu t} g(t + \tau/2)g^*(t - \tau/2) dt$$ (9)

and the Heisenberg parameter [1] $\xi = \frac{1}{4\pi^2 \Delta f \Delta t} \leq 1$ where

$$\left\{ \begin{array}{l} (\Delta t)^2 = \int_{\mathbb{R}} t^2 |g(t)|^2 dt \\ (\Delta f)^2 = \int_{\mathbb{R}} f^2 |G(f)|^2 df \end{array} \right.$$ (10)

in which $g(t)$ is assumed to be oriented-center with unity energy [16] for simple expression. $\Delta t$ is the mass moment of inertia of the prototype function in time and $\Delta f$ in frequency, which shows how the energy (mass) of the prototype function spreads over the time and frequency plane. According to the Heisenberg uncertainty inequality [14], $0 \leq \xi \leq 1$, where the upper bound $\xi = 1$ is achieved by the Gaussian function and the lower bound $\xi = 0$ is achieved by the rectangular function whose $\Delta f$ is infinite. The larger $\xi$ is, the better joint time-frequency localization the prototype function has.

III. SYSTEM IMPLEMENTATION

Rather than deriving the implementation structure from filter banks theory, like in [9]–[11], we try to find the implementation method by directly discretizing the continuous time model without considering the perfect reconstruction (PR) condition. Let $s(t)$ be the output signal of OFDM/QAM modulator

$$s(t) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{N-1} (a_{m,n}^{R}g_{m,2n}(t) + a_{m,n}^{3}g_{m,2n+1}(t))$$ (11)

and the demodulated signal at branch $k$ during symbol duration $n$ can be written as

$$\hat{a}_{m,n}^{R} = \Re \left\{ \int_{\mathbb{R}} s(t)g_{m,2n}(dt) \right\}$$

$$\hat{a}_{m,n}^{3} = \Im \left\{ \int_{\mathbb{R}} s(t)g_{m,2n+1}(dt) \right\}$$ (12)

where $\Re$ and $\Im$ indicate the real and imaginary part respectively.

By sampling $s(t)$ at rate $1/T_c$ during time interval $[nT_s - \tau_0, nT_s + \tau_0)$, we get

$$s(nT_s + kT_c) = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} [a_{m,n}^{R}g_{m,2n}(nT_s + kT_c - lT_s) + ja_{m,n}^{3}g_{m,2n+1}(nT_s + kT_c - lT_s)] e^{j2\pi (m+2l)k T_c}$$ (13)

where $n \in \mathbb{Z}$ and $k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$. Let $s_k[n] = s[nN+k] = s(nT_s + kT_c)$, (13) can be rewritten as

$$s_k[n] = \sum_{p} g(pT_s + kT_c) \left\{ \sum_{m=0}^{N-1} [a_{m,N-p}^{R}e^{j2\pi (m+2n-2p)}e^{j2\pi \frac{k}{T_c}} + ja_{m,N-p}^{3}e^{j2\pi (m+2n-2p)}e^{j2\pi \frac{k}{T_c}}] \right\}$$ (14)

where

$$A_N(x_{m,n}) = \sum_{m=0}^{N-1} x_{m,n} e^{j2\pi (m+2n)}e^{j2\pi \frac{k}{T_c}}$$ (15)

$$g_k[p] = g[pN+k] = g(pT_s + kT_c)$$ (16)

Therefore the OFDM/QAM modulator can be easily implemented by an IFFT block defined in (15) followed by a pulse shaping filter bank defined in (16).

At the receiver side, we sample the received signal $r(t)$ at rate $1/T_c$, and rewrite (12) via approximation as follows

$$\hat{a}_{m,n}^{R} \approx \Re \left\{ T_c \sum_{l=-\infty}^{\infty} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} r(lT_s + kT_c)g_{m,2n}(lT_s + kT_c) \right\}$$

$$\approx \Re \left\{ T_c e^{-j\frac{\pi}{4}(m+2n)} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} r_k[n] * g_k[-n]e^{-j2\pi \frac{k}{T_c}} \right\}$$

$$\hat{a}_{m,n}^{3} \approx \Im \left\{ T_c e^{-j\frac{\pi}{4}(m+2n)} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} r_k[n] * g_k[-n]e^{-j2\pi \frac{k}{T_c}} \right\}$$
where \( g_k[-n] = g[-nN + k] = g(kT_s - NT_k) \). Similarly, the OFDM/OQAM demodulator can be implemented by filter banks \( g_k[n] \) and \( g_k[-n] \) followed by a FFT block.

Assume the pulse shape prototype function \( g(t) \) (or its truncation) has finite duration in \( -M \tau_0 \leq t < M \tau_0 \), its discrete version \( g[n] \) is not empty when \( n = -MN/2, \ldots, MN/2 - 1 \). Therefore the length of each branch filter equals to \( M \).

IV. NUMERICAL RESULTS

A. Time Frequency Localization (TFL)

To illustrate how the demodulation gain varies with respect to the time and frequency spread, the ambiguity function of the output of one demodulation branch is plotted both in a three dimensional plot as well as its two dimensional contour plot, as shown in Fig. 2 which axes are normalized by \( \tau_0 \) and \( \nu_0 \) respectively. Here the data transmitted on each basis function is ignored for simplicity. IOTA prototype function is used and only the neighboring lattice points in the same subset are considered. Those pulses on lattice points with distance \( 2\tau_0 \) or \( 2\nu_0 \) have negative envelope due to the phase factor \( e^{j2\pi(\alpha n + \beta m)} \) which equals to \(-1\) when either \( |n| \) or \( |m| \) equals to 2, but not both. 0 is achieved at the boundary of each lattice grid and therefore no interference will be introduced by neighbors as long as the normalized time or frequency dispersion is less than 2.

The Heisenberg parameter \( \xi \) for each pulse is calculated with two different set of parameters using 32 Samples per normalized time or frequency unit.

<table>
<thead>
<tr>
<th>( t, f \in )</th>
<th>Rect(^a)</th>
<th>HalfCosine</th>
<th>Gauss</th>
<th>IOTA</th>
<th>EGF(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>([-6, 6])</td>
<td>0.3457</td>
<td>0.8949</td>
<td>1.000</td>
<td>0.9769</td>
<td>0.7010</td>
</tr>
<tr>
<td>([-40, 40])</td>
<td>0.1016</td>
<td>0.8911</td>
<td>1.000</td>
<td>0.9769</td>
<td>0.6876</td>
</tr>
</tbody>
</table>

\(^a\) for rectangular pulse, \( (\Delta f)^2 = \int f^2 \text{sinc}^2(\omega f) \text{df} = \infty \) and therefore \( \xi = 0 \) in theory.

\(^b\) for EGF pulse, \( \xi(\alpha) = \xi(1/\alpha) \) and it will steadily increase to its maximum as \( \alpha \) approaches 1 from either direction.

The Gauss pulse achieves the maximum of \( \xi \) and therefore has the best TFL property. The IOTA pulse shows satisfied localization which is the maximum of \( \xi \) among these EGF functions [5]. One thing has to be noticed is that the IOTA prototype function will not be used in our implementation as we have to set \( \tau_0 = T_s/2 \) and \( \nu_0 = F_s/N \), rather than what is demanded in IOTA function where \( \frac{T_s}{\nu_0} = \frac{T_s}{\psi} = 1/\sqrt{2} \).

B. Simulation on SDR Workbench

We have implemented the pulse shaping OFDM/OQAM system in the Matlab/Octave simulation workbench [15]. The FFT/IFFT size is 64 in all the following simulation results. As stated in Section III, the pulse shape prototype function \( g(t) \) is truncated (if necessary) to a finite duration \(-M \tau_0 \leq t < M \tau_0 \).

Fig. 3 presents the signal constellation at the OFDM/OQAM demodulator output when the Half cosine prototype function and the EGF functions are used. Here we choose \( M = 5 \) and therefore the filters on each branch has only \( M = 5 \) nonzero taps. When there is no noise or distortion, the EGF prototype...
can achieve almost perfect reconstruction (see Fig. 3b) while the Half Cosine prototype will result in some distortion (see Fig. 3a). When a random frequency offset ($\Delta f = 0.0001$, i.e. $\Delta f = 1000Hz$) is introduced, as shown in Fig. 3c and Fig. 3d, the distortion in the resulted constellation is notable in both cases and it’s hard to tell which performs better.

For the EGF prototype function, two parameters will affect its performance. One is $\alpha$ and the other is the number of filter taps $M$. Fig. 4 displays the influence of these two parameters in which two different number ($M = 3$, 6) of filter taps and $\alpha = 1$, 3.774 are used respectively. It shows that when the number of filter taps is large enough (e.g. $M = 6$), $\alpha$ will determine the overall performance. The most suitable TFL shape ($\alpha = 1$ when no distortion added) will provide the best performance. While when the number of filter taps is insufficient (e.g. $M = 3$), the most centralized prototype (with highest $\alpha$) will be least affected by truncation (cf. Fig. 4b vs. Fig. 4d).

V. Conclusions

The time frequency localization properties indicated by the Heisenberg parameter, the Ambiguity function, as well as the interference function and the instantaneous correlation functions [16] provide an intuitive way to describe how the signal from different carriers and different symbols get along with one another. As the transmitted signal composed by basis functions will place a copy of the prototype function on each lattice point in the time-frequency plane, the less power the prototype function spreads to the neighboring lattice region, the better reconstruction of the transmitted signal can be retrieved after demodulation.

By adaptively exploiting different prototype functions with diverse TFL property, dynamic spectrum allocation can be achieved in a more natural way, since the transmitter and receiver adapts dynamically to different channel conditions and interference environments so that higher reliability and spectral efficiency can be expected. Also simplified synchronization can be expected as less sensitivity to time and frequency offset. Therefore OFDM/OQAM system is a promising candidate in there future wireless communication.

References